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Completion of L-topological groups

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Abstract

The goal of this paper is to extend an L-topological group to a complete L-topological group, which necessitates formalizing the completion of an L-topological group. In so doing, we introduce the notion of the completion of an L-uniform space in the sense of Gähler, Bayoumi, Kandil and Nouh.

Keywords: L-topological groups; Complete L-topological groups; L-uniform spaces; Complete L-uniform spaces; L-filters; $\mathscr{U}$-Cauchy filters; L-topological spaces

1. Introduction

In this paper, we give new notions for L-filters in the sense of [13], L-uniform spaces in the sense of [15] and L-topological groups in the sense of [3], respectively, called *$\mathcal{U}$-Cauchy filter*, *complete L-uniform space* and *complete L-topological group*. We define $\mathcal{U}$-Cauchy filters for L-uniform spaces, characterize complete L-uniform spaces, and subsequently characterize complete L-topological groups. Completions of L-uniform spaces and L-topological groups are characterized and studied. Many important theorems of the classical theory of uniform spaces and topological groups are, respectively, extended to L-uniform spaces and L-topological groups.

In Section 2, we recall some results on L-filters and L-neighborhood filters defined by Gähler in [11,13,14]. We also define the product of two L-sets and the product of two L-filters.

The notion of $\mathcal{U}$-Cauchy filter in an L-uniform space $(X, \mathcal{U})$ is introduced and studied in Section 3. We show that any convergent L-filter is a $\mathcal{U}$-Cauchy filter and that the converse holds in complete L-uniform spaces.

Section 4 extends an L-uniform space to a complete L-uniform space. The completion of an L-uniform space is given as a reduced extension of an L-uniform space with a complete L-uniform structure.

The notion of the completion of an L-topological group is introduced in Section 5. Using the bilateral L-uniform structure $\mathcal{U}^{\mathrm{b}} = \mathcal{U}^{\mathrm{l}} \vee \mathcal{U}^{\mathrm{r}}$ which is the supremum of the left invariant L-uniform structure $\mathcal{U}^{\mathrm{l}}$ and the right invariant L-uniform structure $\mathcal{U}^{\mathrm{r}}$ of an L-topological group (G, τ) as defined in [8], we define the notion of a complete L-topological group. A complete separated L-topological group (H, σ) in which (G, τ) is a dense subgroup will be called a completion of (G, τ).

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There are other filter approaches to uniform space in the fuzzy case. Gutiérrez Garcia et al. introduced in [17] a unification to these approaches using *L*-filters. In our filter approach to fuzzy uniform spaces we use a notion of *L*-filter different from that given in [17] (more precisely, our condition (F1) is different from [17]). So, our approach is not the same as [17]. A deeper comparison could be done in future work.

2. On *L*-filters

In this section, we recall and prove some results concerning *L*-filters needed in the paper. Denote by L^X the set of all *L*-subsets of a non-empty set *X*, where *L* is a complete chain with different least and greatest elements 0 and 1, respectively [16]. We may note that $L_0 = L \setminus \{0\}$. For each *L*-set $\lambda \in L^X$, let λ' denote the complement of λ, defined by $\lambda'(x) = \lambda(x)'$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_0$, the *L*-subset x_α of *X* with value α at *x* and 0 otherwise is called an *L*-it point in *X* and the *L*-subset of *X* with constant value α will be denoted by $\overline{\alpha}$.

2.1. L-filters

By an *L-filter* on a non-empty set *X* we mean [13] a mapping $\mathcal{M} : L^X \to L$ such that $\mathcal{M}(\overline{\alpha}) \leqslant \alpha$ for all $\alpha \in L$, $\mathcal{M}(\overline{1}) = 1$, and $\mathcal{M}(\lambda \wedge \mu) = \mathcal{M}(\lambda) \wedge \mathcal{M}(\mu)$ for all $\lambda, \mu \in L^X$. $\mathcal{M}$ ia called *homogeneous* [11] if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are *L*-filters on *X*, $\mathcal{M}$ is called *finer* than $\mathcal{N}$, denoted by $\mathcal{M} \leqslant \mathcal{N}$, provided $\mathcal{M}(\lambda) \geqslant \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$.

Let $\mathcal{F}_L X$ denote the set of all *L*-filters on *X*, $f : X \to Y$ a mapping and $\mathcal{M}, \mathcal{N}$ are *L*-filters on *X*, *Y*, respectively. Then the *image* of $\mathcal{M}$ and the *preimage* of $\mathcal{N}$ with respect to *f* are the *L*-filters $\mathcal{F}_L f(\mathcal{M})$ on *Y* and $\mathcal{F}_L^- f(\mathcal{N})$ on *X* defined by $\mathcal{F}_L f(\mathcal{M})(\mu) = \mathcal{M}(\mu \circ f)$ for all $\mu \in L^Y$ and $\mathcal{F}_L^- f(\mathcal{N})(\lambda) = \bigvee_{\mu \circ f \leqslant \lambda} \mathcal{N}(\mu)$ for all $\lambda \in L^X$, respectively. For each mapping $f : X \to Y$ and each *L*-filter $\mathcal{N}$ on *Y*, for which the preimage $\mathcal{F}_L^- f(\mathcal{N})$ exists, we have $\mathcal{F}_L f(\mathcal{F}_L^- f(\mathcal{N})) \leqslant \mathcal{N}$. Moreover, for each *L*-filter $\mathcal{M}$ on *X*, the inequality $\mathcal{M} \leqslant \mathcal{F}_L^- f(\mathcal{F}_L f(\mathcal{M}))$ holds [13].

For each non-empty set *A* of *L*-filters on *X*, the supremum $\bigvee_{\mathcal{M} \in A} \mathcal{M}$ with respect to the finer relation of *L*-filters exists and we have

$$\left(\bigvee_{\mathcal{M} \in A} \mathcal{M} \right)(f) = \bigwedge_{\mathcal{M} \in A} \mathcal{M}(f)$$

for all $f \in L^X$. The infimum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$ of *A* exists *if and only if* for each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of *A* we have $\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n) \leqslant \sup(\lambda_1 \wedge \cdots \wedge \lambda_n)$ for all $\lambda_1, \ldots, \lambda_n \in L^X$ [11]. If the infimum of *A* exists, then for each $\lambda \in L^X$ and *n* a positive integer we have

$$\left(\bigwedge_{\mathcal{M} \in A} \mathcal{M} \right)(\lambda) = \bigvee_{\substack{\lambda_1 \wedge \cdots \wedge \lambda_n \leqslant \lambda, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in A}} (\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n)).$$

By a *filter* on *X* we mean a non-empty subset $\mathcal{F}$ of L^X which does not contain $\overline{0}$ and closed under finite infima and super sets [18]. For each *L*-filter $\mathcal{M}$ on *X*, the subset α-pr $\mathcal{M}$ of L^X defined by: $\alpha\text{-pr}\, \mathcal{M} = \{\lambda \in L^X | \mathcal{M}(\lambda) \geqslant \alpha\}$ is a filter on *X*.

A family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X is called a *valued L-filter base* on *X* [13] if the following conditions are fulfilled:

(V1) $\lambda \in \mathcal{B}_\alpha$ implies $\alpha \leqslant \sup \lambda$.
(V2) For all $\alpha, \beta \in L_0$ and all *L*-sets $\lambda \in \mathcal{B}_\alpha$ and $\mu \in \mathcal{B}_\beta$, if $\alpha \wedge \beta > 0$ holds, then there are a $\gamma \geqslant \alpha \wedge \beta$ and an *L*-set $\nu \leqslant \lambda \wedge \mu$ such that $\nu \in \mathcal{B}_\gamma$.

Each valued *L*-filter base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ on a set *X* defines an *L*-filter $\mathcal{M}$ on *X* by $\mathcal{M}(\lambda) = \bigvee_{\mu \in \mathcal{B}_\alpha,\, \mu \leqslant \lambda} \alpha$ for all $\lambda \in L^X$. On the other hand, each *L*-filter $\mathcal{M}$ can be generated by many valued *L*-filter bases, and among them the greatest one is $(\alpha\text{-pr}\, \mathcal{M})_{\alpha \in L_0}$. We may note that *L*-filters are called fuzzy filters in [13–15] and also filters are called prefilters in [16].

Proposition 2.1 (*Gähler [13]*). *There is a one-to-one correspondence between the L- filters $\mathcal{M}$ on X and the families $(\mathcal{M}_\alpha)_{\alpha\in L_0}$ of filters on X which fulfill the following conditions*:

(1) $f \in \mathcal{M}_\alpha$ *implies* $\alpha \leqslant \sup f$.
(2) $0 < \alpha \leqslant \beta$ *implies* $\mathcal{M}_\alpha \supseteq \mathcal{M}_\beta$.
(3) *For each* $\alpha \in L_0$ *with* $\bigvee_{0<\beta<\alpha} \beta = \alpha$ *we have* $\bigcap_{0<\beta<\alpha} \mathcal{M}_\beta = \mathcal{M}_\alpha$.

This correspondence is given by $\mathcal{M}_\alpha = \alpha\text{-pr}\,\mathcal{M}$ *for all* $\alpha \in L_0$ *and* $\mathcal{M}(f) = \bigvee_{g\in\mathcal{M}_\alpha,\, g\leqslant f} \alpha$ *for all* $f \in L^X$.

2.2. L-neighborhood filters

In the following, the topology in sense of [10,16] will be used which will be called *L*-topology. int_τ and cl_τ denote the interior and the closure operators with respect to the *L*-topology τ, respectively. For each *L*-topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by $\mathcal{N}(x)(\lambda) = \text{int}_\tau\, \lambda(x)$ for all $\lambda \in L^X$ is an *L*-filter on *X*, called the *L-neighborhood filter* of the space (X, τ) at *x*, and for short is called a τ-neighborhood filter at *x*. The mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\lambda) = \lambda(x)$ for all $\lambda \in L^X$ is a homogeneous *L*-filter on *X*. Let (X, τ) and (Y, σ) be two *L*-topological spaces. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ is called *L-continuous* (or (τ, σ)-*continuous*) provided $\text{int}_\sigma\, \mu \circ f \leqslant \text{int}_\tau\, (\mu \circ f)$ for all $\mu \in L^Y$. An *L*-filter $\mathcal{M}$ is said to *converge to* $x \in X$, denoted by $\mathcal{M} \underset{\tau}{\to} x$, if $\mathcal{M} \leqslant \mathcal{N}(x)$ [14]. The *L*-neighborhood filter $\mathcal{N}(F)$ at an ordinary subset *F* of *X* is the *L*-filter on *X* defined by the authors in [5], by means of $\mathcal{N}(x)$, $x \in F$ as

$$\mathcal{N}(F) = \bigvee_{x\in F} \mathcal{N}(x).$$

The *L*-filter $\dot{F}$ is defined by $\dot{F} = \bigvee_{x\in F} \dot{x}$. $\dot{F} \leqslant \mathcal{N}(F)$ holds for all $F \subseteq X$.

Lemma 2.1 (*Gähler [14]*). *Let* (X, τ) *and* (Y, σ) *be two L-topological spaces and* $\mathcal{M}$ *an L-filter on X. If* $f : X \to Y$ *is a* (τ, σ)-*continuous mapping, then* $\mathcal{M} \underset{\tau}{\to} x$ *implies that* $\mathcal{F}_L f(\mathcal{M}) \underset{\sigma}{\to} f(x)$ *holds.*

To define the product of two *L*-filters, first we need to define the product of two *L*-sets.

For any $\lambda, \mu \in L^X$, let $\lambda \times \mu : X \times X \to L$ be the *L*-set defined as follows:

$$(\lambda \times \mu)(x, y) \;=\; \lambda(x) \wedge \mu(y) \tag{2.1}$$

for all $x, y \in X$.

Remark 2.1. *For all* $\lambda, \mu, \xi, \eta \in L^X$, *we have*

$$(\lambda \wedge \mu) \times (\xi \wedge \eta) = (\lambda \times \xi) \wedge (\mu \times \eta) = (\lambda \times \eta) \wedge (\mu \times \xi).$$

The following proposition introduces the product of two *L*-filters.

Proposition 2.2. *For any two L-filters* $\mathcal{L}, \mathcal{M}$ *on X, the mapping* $\mathcal{L} \times \mathcal{M} : L^{X\times X} \to L$ *defined by*

$$(\mathcal{L} \times \mathcal{M})(u) = \bigvee_{\lambda\times\mu\leqslant u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \tag{2.2}$$

for all $u \in L^{X\times X}$ *is an L-filter on* $X \times X$, *called* the product *L*-filter of $\mathcal{L}$ and $\mathcal{M}$.

Proof. From (2.1) and that $\mathcal{L}, \mathcal{M}$ are *L*-filters, we get

$$(\mathcal{L} \times \mathcal{M})(\widetilde{\alpha}) = \bigvee_{\lambda\times\mu\leqslant\widetilde{\alpha}} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \leqslant \alpha.$$

Moreover, $(\mathcal{L} \times \mathcal{M})(\widetilde{1}) = 1$.

From Remark 2.1 and for all $u, v \in L^{X\times X}$, we get

$$\begin{aligned}(\mathcal{L}\times\mathcal{M})(u)\wedge(\mathcal{L}\times\mathcal{M})(v) &= \bigvee_{\lambda\times\mu\leqslant u}(\mathcal{L}(\lambda)\wedge\mathcal{M}(\mu))\wedge\bigvee_{\xi\times\eta\leqslant v}(\mathcal{L}(\xi)\wedge\mathcal{M}(\eta))\\ &= \bigvee_{\lambda\times\mu\leqslant u,\,\xi\times\eta\leqslant v}(\mathcal{L}(\lambda\wedge\xi)\wedge\mathcal{M}(\mu\wedge\eta))\\ &\leqslant \bigvee_{(\lambda\wedge\xi)\times(\mu\wedge\eta)\leqslant u\wedge v}(\mathcal{L}(\lambda\wedge\xi)\wedge\mathcal{M}(\mu\wedge\eta))\\ &= (\mathcal{L}\times\mathcal{M})(u\wedge v).\end{aligned}$$

Also,

$$\begin{aligned}(\mathcal{L}\times\mathcal{M})(u\wedge v) &= \bigvee_{\lambda\times\mu\leqslant u\wedge v}(\mathcal{L}(\lambda)\wedge\mathcal{M}(\mu)\\ &\leqslant \bigvee_{\lambda\times\mu\leqslant u,\,\lambda\times\mu\leqslant v}(\mathcal{L}(\lambda)\wedge\mathcal{M}(\mu))\\ &= \bigvee_{\lambda\times\mu\leqslant u}(\mathcal{L}(\lambda)\wedge\mathcal{M}(\mu))\wedge\bigvee_{\lambda\times\mu\leqslant v}(\mathcal{L}(\lambda)\wedge\mathcal{M}(\mu))\\ &= (\mathcal{L}\times\mathcal{M})(u)\wedge(\mathcal{L}\times\mathcal{M})(v).\end{aligned}$$

Hence, $(\mathcal{L}\times\mathcal{M})$ is an L-filter on $X\times X$. □

The product of two L-filters provides the following result.

Lemma 2.2. *Let $\mathcal{L}$ and $\mathcal{M}$ be L-filters on X, and let $(\mathcal{L}_\alpha)_{\alpha\in L_0}$ and $(\mathcal{M}_\alpha)_{\alpha\in L_0}$ be the families of filters on X correspond, according to Proposition* 2.1, *to $\mathcal{L}$ and $\mathcal{M}$, respectively. Then the family $(\mathcal{K}_\alpha)_{\alpha\in L_0}$ of subsets $\mathcal{K}_\alpha$ of $L^{X\times X}$, where*

$$\mathcal{K}_\alpha = \{\lambda\times\mu|\lambda\in\mathcal{L}_\alpha,\ \mu\in\mathcal{M}_\alpha\} \tag{2.3}$$

is a family of filters on $X\times X$ which corresponds to the product L-filter $\mathcal{L}\times\mathcal{M}$.

Proof. Since $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$ are non-empty subsets of L^X for all $\alpha\in L_0$, then also $\mathcal{K}_\alpha = \{\lambda\times\mu|\lambda\in\mathcal{L}_\alpha,\ \mu\in\mathcal{M}_\alpha\}$ is non-empty for any $\alpha\in L_0$. Also, $\overline{0}$ does not exist in $\mathcal{L}_\alpha$ or $\mathcal{M}_\alpha$ implies that $\overline{0}\notin\mathcal{K}_\alpha$ for all $\alpha\in L_0$. From Remark 2.1 and from the fact that $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$ are filters, for all $u, v\in\mathcal{K}_\alpha$ and $w\geqslant v$ we get $u\wedge v\in\mathcal{K}_\alpha$ and $w\in\mathcal{K}_\alpha$ for all $\alpha\in L_0$. That is, $\mathcal{K}_\alpha$, for all $\alpha\in L_0$, is a filter on $X\times X$.

Let $u\in\mathcal{K}_\alpha$. Then $u=\lambda\times\mu$, where $\lambda\in\mathcal{L}_\alpha$ and $\mu\in\mathcal{M}_\alpha$. From condition (1) for $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$, we get $\alpha\leqslant\sup\lambda$ and $\alpha\leqslant\sup\mu$ and then $\alpha\leqslant\sup(\lambda\times\mu)=\sup u$. Hence, $\mathcal{K}_\alpha$ provides condition (1) of Proposition 2.1.

Let $0<\alpha\leqslant\beta$ and $u\in\mathcal{K}_\beta$. Then $u=\lambda\times\mu$ for $\lambda\in\mathcal{L}_\beta$ and $\mu\in\mathcal{M}_\beta$. Since $\mathcal{L}_\alpha\supseteq\mathcal{L}_\beta$ and $\mathcal{M}_\alpha\supseteq\mathcal{M}_\beta$, then $\lambda\in\mathcal{L}_\alpha$ and $\mu\in\mathcal{M}_\alpha$ and hence $u\in\mathcal{K}_\alpha$. That is, $\mathcal{K}_\alpha$ fulfills the requirements of condition (2) of Proposition 2.1.

Since $\bigcap_{0<\beta<\alpha}\mathcal{L}_\beta=\mathcal{L}_\alpha$ and $\bigcap_{0<\beta<\alpha}\mathcal{M}_\beta=\mathcal{M}_\alpha$, we get that

$$\begin{aligned}\bigcap_{0<\beta<\alpha}\mathcal{K}_\beta &= \bigcap_{0<\beta<\alpha}\{\lambda\times\mu|\lambda\in\mathcal{L}_\beta,\mu\in\mathcal{M}_\beta\}\\ &= \left\{\lambda\times\mu|\lambda\in\bigcap_{0<\beta<\alpha}\mathcal{L}_\beta,\mu\in\bigcap_{0<\beta<\alpha}\mathcal{M}_\beta\right\}\\ &= \{\lambda\times\mu|\lambda\in\mathcal{L}_\alpha,\mu\in\mathcal{M}_\alpha\}\\ &= \mathcal{K}_\alpha.\end{aligned}$$

This means that condition (3) of Proposition 2.1 also holds for $\mathcal{K}_\alpha$.

Hence, according to Proposition 2.1, we get a one-to-one correspondence between the family $(\mathcal{K}_\alpha)_{\alpha\in L_0}$ of the filters on $X \times X$ defined by (2.3) and the product L-filter $\mathcal{L} \times \mathcal{M}$ on $X \times X$ defined by (2.2). Therefore,

$$(\mathcal{L} \times \mathcal{M})(u) = \bigvee_{v\in\mathcal{K}_\alpha,\, v\leqslant u} \alpha \quad \text{and} \quad \alpha\text{-pr}\,(\mathcal{L} \times \mathcal{M}) = \mathcal{K}_\alpha$$

for all $u \in L^{X\times X}$ and for all $\alpha \in L_0$. □

3. $\mathcal{U}$-Cauchy filters

This section is devoted to study a notion of Cauchy filter on L-uniform spaces as defined in [15].

3.1. L-uniform spaces

An L-filter $\mathcal{U}$ on $X \times X$ is called an L-uniform structure on X [15] if the following conditions are fulfilled:

(U1) $(x, x)^{\bullet} \leqslant \mathcal{U}$ for all $x \in X$;
(U2) $\mathcal{U} = \mathcal{U}^{-1}$;
(U3) $\mathcal{U} \circ \mathcal{U} \leqslant \mathcal{U}$.

Where $(x, x)^{\bullet}(u) = u(x, x)$, $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ and $(\mathcal{U} \circ \mathcal{U})(u) = \bigvee_{v\circ w\leqslant u} \mathcal{U}(v \wedge w)$ for all $u \in L^{X\times X}$, and $u^{-1}(x, y) = u(y, x)$ and $(v \circ w)(x, y) = \bigvee_{z\in X}(w(x, z) \wedge v(z, y))$ for all $x, y \in X$.

A set X equipped with an L-uniform structure $\mathcal{U}$ is called an *L-uniform space*.

To each L-uniform structure $\mathcal{U}$ on X is associated a stratified L-topology $\tau_{\mathcal{U}}$ whose interior operator is given by

$$(\text{int}_{\mathcal{U}}\, \lambda)(x) = \mathcal{U}[\dot{x}](\lambda)$$

for all $x \in X$ and all $\lambda \in L^X$, where $\mathcal{U}[\dot{x}](\lambda) = \bigvee_{u[\mu]\leqslant\lambda} (\mathcal{U}(u) \wedge \mu(x))$ and $u[\mu](x) = \bigvee_{y\in X} (\mu(y) \wedge u(y, x))$. For all $x \in X$ we have

$$\mathcal{U}[\dot{x}] = \mathcal{N}(x),$$

where $\mathcal{N}(x)$ is the L-neighborhood filter of the space $(X, \tau_{\mathcal{U}})$ at x. That is, an L-filter $\mathcal{M}$ on an L-uniform space $(X, \mathcal{U})$ is said to converge to $x \in X$ if $\mathcal{M} \leqslant \mathcal{U}[\dot{x}]$.

Let $\mathcal{U}$ be an L-uniform structure on a set X. Then $u \in L^{X\times X}$ is called a *surrounding* provided $\mathcal{U}(u) \geqslant \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$ [8].

Given a surrounding u in $(X, \mathcal{U})$, a subset $A \subseteq X$ is called *small of order u* if $u(x, y) \geqslant \alpha$ for all $x, y \in A$ and for some $\alpha \in L_0$.

Definition 3.1. In an L-uniform space $(X, \mathcal{U})$, an L-filter $\mathcal{M}$ on X is said to be a *$\mathcal{U}$-Cauchy filter* provided that for any surrounding u, there exists a set $B \subseteq X$ such that $\mathcal{M} \leqslant \dot{B}$ and B is small of order u.

Now, we prove the following expected result for the convergent L-filters.

Proposition 3.1. *Every convergent L-filter on an L-uniform space $(X, \mathcal{U})$ is a $\mathcal{U}$-Cauchy filter.*

Proof. Let $\mathcal{M}$ be an L-filter on X which converges to $x \in X$, that is, $\mathcal{M} \leqslant \mathcal{U}[\dot{x}]$. Then we can choose a set $B \subseteq X$ such that $\mathcal{M} \leqslant \dot{B} = \mathcal{U}[\dot{x}]$ which means

$$\mathcal{M}(\lambda) \geqslant \bigvee_{u[\mu]\leqslant\lambda} (\mathcal{U}(u) \wedge \mu(x)) = \bigwedge_{y\in B} \lambda(y) = \dot{B}(\lambda)$$

for all $\lambda \in L^X$. Since $(x, x)^{\bullet} \leqslant \mathcal{U}$ for all $x \in X$, then $u(x, x) \geqslant \mathcal{U}(u) \geqslant \alpha$ for any surrounding u and for some $\alpha \in L_0$. Now, $x \in B$ implies that $\dot{x} \leqslant \mathcal{U}[\dot{x}] = \dot{B}$ holds. Also, for any $y \in B$ we get $\bigvee_{u[\mu]\leqslant\lambda} (\alpha \wedge \mu(x)) \leqslant \lambda(y)$, and consequently $\bigvee_z (u(z, y) \wedge \mu(z)) \leqslant \lambda(y)$, and so $\alpha \wedge \mu(x) \leqslant u(x, y) \wedge \mu(x) \leqslant \lambda(y)$. Thus, for all $x, y \in B$, we have $u(x, y) \geqslant \alpha$ for some $\alpha \in L_0$ and $\mathcal{M} \leqslant \dot{B}$. Hence, there is a set $B \subseteq X$ which is small of order any surrounding u in $(X, \mathcal{U})$ and $\mathcal{M} \leqslant \dot{B}$, and therefore $\mathcal{M}$ is a $\mathcal{U}$-Cauchy filter on X. □

Let A be a subset of a set X, $\mathcal{U}$ an L-uniform structure on X and $i : A \hookrightarrow X$ the inclusion mapping of A into X. Then the initial L-uniform structure $\mathcal{F}_L^{-}(i \times i)(\mathcal{U})$ of $\mathcal{U}$ with respect to i, denoted by $\mathcal{U}_A$, is called an *L-uniform substructure* of $\mathcal{U}$ and $(A, \mathcal{U}_A)$ an *L-uniform subspace* of $(X, \mathcal{U})$ [2].

We have the following result.

Lemma 3.1. *Let $(X, \mathcal{U})$ be an L-uniform space and A a non-empty subset of X. Then an L-filter on A is a $\mathcal{U}_A$-Cauchy filter* if and only if *it is a $\mathcal{U}$-Cauchy filter*.

Proof. Let $\mathcal{M}$ be a $\mathcal{U}_A$-Cauchy filter on A. Then there exists $B \subseteq A$ with $\mathcal{M} \leqslant \dot{B}$ and B small of order any surrounding u_A in $(A, \mathcal{U}_A)$. This means that there is $B \subseteq A \subseteq X$ such that $\mathcal{M} \leqslant \dot{B}$ and $u_A(x, y) \geqslant \alpha$ for all $x, y \in B$ and for some $\alpha \in L_0$. That is, for any surrounding u in $(X, \mathcal{U})$,

$$u(x, y) = (u \circ (i \times i))(x, y) = u_A(x, y) \geqslant \alpha$$

for all $x, y \in B$ and for some $\alpha \in L_0$, and then $\mathcal{M} \leqslant \dot{B}$ and $B \subseteq X$ is small of order any surrounding u in $(X, \mathcal{U})$. Hence, $\mathcal{M}$ is a $\mathcal{U}$-Cauchy filter.

Conversely, if $\mathcal{M}$ is a $\mathcal{U}$-Cauchy filter on X, then there exists $B \subseteq A \subseteq X$ with $\mathcal{M} \leqslant \dot{B}$ and B is small of order any surrounding u in $(X, \mathcal{U})$. Hence, $u(x, y) \geqslant \alpha$ for all $x, y \in B$ and for some $\alpha \in L_0$, which means that, for every surrounding u_A in $(A, \mathcal{U}_A)$,

$$u_A(x, y) = (u \circ (i \times i))(x, y) = u(x, y) \geqslant \alpha$$

for all $x, y \in B$ and for some $\alpha \in L_0$. Therefore, $\mathcal{M} \leqslant \dot{B}$ and $B \subseteq A$ is small of order any surrounding u_A in $(A, \mathcal{U}_A)$, that is, $\mathcal{M}$ is a $\mathcal{U}_A$-Cauchy filter. □

A mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ between L-uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is said to be *L-uniformly continuous* (or $(\mathcal{U}, \mathcal{V})$-*continuous*) provided that

$$\mathcal{F}_L(f \times f)(\mathcal{U}) \leqslant \mathcal{V}$$

holds.

In the next sections we shall use the following result.

Lemma 3.2. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be L-uniform spaces and $f : X \to Y$ a $(\mathcal{U}, \mathcal{V})$-continuous mapping. If $\mathcal{M}$ is a $\mathcal{U}$-Cauchy filter, then $\mathcal{F}_L f(\mathcal{M})$ is a $\mathcal{V}$-Cauchy filter*.

Proof. $\mathcal{M}$ is a $\mathcal{U}$-Cauchy filter on X means that there exists $B \subseteq X$ such that $\mathcal{M} \leqslant \dot{B}$ and B is small of order any surrounding u in $(X, \mathcal{U})$. This means that $\mathcal{M} \leqslant \dot{B}$ and $u(x, y) \geqslant \alpha$ for all $x, y \in B$ and for some $\alpha \in L_0$ which implies that

$$\mathcal{F}_L f(\mathcal{M}) \leqslant \mathcal{F}_L f(\dot{B}) = (f(\dot{B}))$$

holds for the set $f(B) \subseteq Y$. Let v be a surrounding in $(Y, \mathcal{V})$. Since f is $(\mathcal{U}, \mathcal{V})$-continuous, we have

$$\alpha \leqslant \mathcal{V}(v) \leqslant \mathcal{U}(v \circ (f \times f)) = \mathcal{F}_L(f \times f)(\mathcal{U})(v)$$

for some $\alpha \in L_0$. Since $v = v^{-1}$, then $(v \circ (f \times f))^{-1} = v^{-1} \circ (f \times f) = v \circ (f \times f)$. That is, $u = v \circ (f \times f)$ is a surrounding in $(X, \mathcal{U})$ which means that

$$\alpha \leqslant u(x, y) = (v \circ (f \times f))(x, y) = v(f(x), f(y))$$

for all $f(x), f(y) \in f(B)$ and for some $\alpha \in L_0$. Hence, $\mathcal{F}_L f(\mathcal{M}) \leqslant (f(\dot{B}))$ for the set $f(B) \subseteq Y$ and $f(B)$ is small of order every surrounding in $(Y, \mathcal{V})$. Therefore, $\mathcal{F}_L f(\mathcal{M})$ is a $\mathcal{V}$-Cauchy filter on Y. □

4. Completion of L-uniform spaces

Here, we introduce a notion of the completion of an L-uniform space.

Firstly, some general results.

If (Y, σ) is an L-topological space and X is a non-empty subset of Y, then the initial L-topology of σ, with respect to the inclusion mapping $i : X \hookrightarrow Y$, is the L-topology $i^{-1}(\sigma) = \{i^{-1}(\lambda) | \lambda \in \sigma\}$ on X and is denoted by σ_X.

An L-topological space (Y, σ) is called an *extension* of the L-topological space (X, τ) if $X \subseteq Y$, $\tau = \sigma_X$ and X is σ-dense in Y, that is, $\text{cl}_\sigma X = Y$.

The extension (Y, σ) of (X, τ) is called *reduced* if for any $x \neq y$ in Y and $x \in Y \backslash X$, we have $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$, where $\mathcal{N}_\sigma(x)$ denotes the L-neighborhood filter of (Y, σ) at a point $x \in Y$.

4.1. GT_i-spaces

In [4,5,7], we have introduced and studied the notion of GT_i-spaces for all $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$ as follows.

An L-topological space (X, τ) is called [4,5,7]:

(1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ or $\dot{y} \nleq \mathcal{N}(x)$.
(2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ and $\dot{y} \nleq \mathcal{N}(x)$.
(3) GT_2 if for all $x, y \in X$ with $x \neq y$, we have $\mathcal{M} \nleq \mathcal{N}(x)$ or $\mathcal{M} \nleq \mathcal{N}(y)$ for all L-filters $\mathcal{M}$ on X.
(4) *Regular* if for all $x \notin F$ and $F = \text{cl}_\tau F$, $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist.
(5) GT_3 if it is GT_1 and regular.
(6) *Completely regular* if for all $x \notin F \in \tau'$, there exists an L- continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.
(7) $GT_{3\frac{1}{2}}$ (or *L-Tychonoff*) if it is GT_1 and completely regular.

In the following GT_i-*space* means an L-topological space which is GT_i, $i = 0, 1, 2, 3, 3\frac{1}{2}$.

Proposition 4.1 (*Bayoumi and Ibedou [4,5,7]*). *Every GT_i-space is a GT_{i-1}-space for each $i = 1, 2, 3$, and every $GT_{3\frac{1}{2}}$-space is a GT_3-space.*

Now, we have the following results.

Lemma 4.1. *If the extension (Y, σ) of (X, τ) is a GT_0-space, then (Y, σ) is a reduced extension of (X, τ).*

Proof. Straightforward. □

Lemma 4.2. *Any reduced extension (Y, σ) of a GT_0-space (X, τ) is a GT_0-space.*

Proof. For all $x \neq y$ in $Y \backslash X$, we have $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$. Also for all $x \neq y$ in X, we have $\mathcal{N}_\tau(x) \neq \mathcal{N}_\tau(y)$. Hence, for all $x \neq y$ in Y we get that $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$, and thus (Y, σ) is a GT_0-space. □

Since for all $f, g \in L^X$, $\text{int}_\sigma f(y) = f(x)$ for some $x \in X$ implies that $\text{int}_\sigma f(y) \wedge \bigwedge_{x \in X} g(x) \leqslant f(x)$ holds for some $x \in X$ and also $\text{int}_\sigma f(y) \wedge \bigwedge_{x \in X} g(x) \leqslant g(x)$ holds for all $x \in X$, then $\text{int}_\sigma f(y) \wedge \bigwedge_{x \in X} g(x) \leqslant \sup(f \wedge g)$ for all $f, g \in L^X$. That is, the infimum $\mathcal{N}_\sigma(y) \wedge \dot{X}$ exists. This provides the following.

Remark 4.1. Let (X, τ) be an L-topological space and $X \subseteq Y$. If we succeed in defining an L-topology σ on Y such that (Y, σ) is an extension of (X, τ), then X σ-dense in Y implies that every σ-neighborhood of each $y \in Y$ intersects X, that is, the infimum $\mathcal{N}_\sigma(y) \wedge \dot{X}$ exists.

Definition 4.1. Let (X, τ) and (Y, σ) be two L-topological spaces. If (Y, σ) is an extension of (X, τ), then, for any $x \in Y$, an L-filter $\mathcal{N}_\sigma(x) \wedge \dot{X}$ is said to be a *trace filter at* x.

The filter $\mathcal{N}_\sigma(x) \wedge \dot{X}$ is denoted by $\mathcal{M}_x$. Obviously, $\mathcal{M}_x = \mathcal{N}_\tau(x)$ whenever $x \in X$. It is clear that $\mathcal{M}_x \underset{\sigma}{\to} x$.

Definition 4.2. Let (X, τ) and (Y, σ) be two L-topological spaces, (X', τ^*) an extension of (X, τ) and let $f : X \to Y$ be a (τ, σ)-continuous mapping. Then the restriction mapping $g|_X$ on X of the (τ^*, σ)-continuous mapping $g : X' \to Y$, which coincides with f, is called a *continuous extension* of f into X'.

Remark 4.2. Let (X, τ) and (Y, σ) be two L-topological spaces, (X', τ^*) an extension of (X, τ), $f : X \to Y$ a mapping and $\mathcal{M}_x = \mathcal{N}_{\tau^*}(x) \wedge \dot{X}$ a trace filter on X at $x \in X'$. For the existence of a continuous extension $g : X' \to Y$, it is necessary that f be (τ, σ)-continuous and $\mathcal{F}_L f(\mathcal{M}_x) \underset{\sigma}{\to} x$ for a trace filter $\mathcal{M}_x$ at $x \in X'$. If (Y, σ) is a regular space, then these conditions also are sufficient. It is clear that $\mathcal{M}_x \underset{\sigma}{\to} x$.

Lemma 4.3. *With the notations in Remark* 4.2, *let* $g_1 : X' \to Y$ *and* $g_2 : X' \to Y$ *be* (τ^*, σ)*-continuous. If* (Y, σ) *is a* GT_2*-space and* $g_1|_X = g_2|_X = f$, *then* $g_1 = g_2$.

Proof. Let $x \in X'$ be arbitrary and $\mathcal{M}_x \underset{\tau^*}{\to} x$. From Lemma 2.1, we get $\mathcal{F}_L g_1(\mathcal{M}_x) \underset{\sigma}{\to} g_1(x)$ and $\mathcal{F}_L g_2(\mathcal{M}_x) \underset{\sigma}{\to} g_2(x)$ holds. Also we have $\mathcal{F}_L g_1(\mathcal{M}_x) = \mathcal{F}_L g_2(\mathcal{M}_x) = \mathcal{F}_L f(\mathcal{M}_x)$. Since (Y, σ) is a GT_2-space, then $g_1(x) = g_2(x)$. Thus $g_1 = g_2$. □

Lemma 4.4. *An extension* (Y, σ) *of* (X, τ) *is reduced* if and only if $\mathcal{M}_x \neq \mathcal{M}_y$ *for all* $x \neq y$ *in* Y *and* $x \in Y \backslash X$.

Proof. The proof follows from the fact that

$$\mathcal{M}_x = \mathcal{N}_\sigma(x) \wedge \dot{X} \neq \mathcal{N}_\sigma(y) \wedge \dot{X} = \mathcal{M}_y$$

if and only if $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$. □

Definition 4.3. An L-uniform space $(Y, \mathcal{U}^*)$ is called an *extension* of the L-uniform space $(X, \mathcal{U})$ if $X \subseteq Y, \mathcal{U} = \mathcal{U}^*_X$ and X is a $\tau_{\mathcal{U}^*}$-dense in Y.

Definition 4.4. An L-uniform space $(Y, \mathcal{U}^*)$ is called a *reduced extension* of an L-uniform space $(X, \mathcal{U})$ if $(Y, \tau_{\mathcal{U}^*})$ is a reduced extension of $(X, \tau_{\mathcal{U}})$.

An L-uniform structure $\mathcal{U}$ on a set X is called *separated* [6] if for all $x, y \in X$ with $x \neq y$ there is $u \in L^{X \times X}$ such that $\mathcal{U}(u) = 1$ and $u(x, y) = 0$. The space $(X, \mathcal{U})$ is called a *separated L-uniform space*.

Proposition 4.2 (*Bayoumi and Ibedou [6]*). *Let X be a set,* $\mathcal{U}$ *an L-uniform structure on X and* $\tau_{\mathcal{U}}$ *the L-topology associated with* $\mathcal{U}$. *Then* $(X, \mathcal{U})$ *is separated if and only if* $(X, \tau_{\mathcal{U}})$ *is* GT_0*-space.*

Lemma 4.5. *If* $(X, \mathcal{U})$ *is a separated L-uniform space and* $(Y, \mathcal{U}^*)$ *is a reduced extension of* $(X, \mathcal{U})$, *then* $(Y, \mathcal{U}^*)$ *is separated as well.*

Proof. From Proposition 4.2, we get $(X, \tau_{\mathcal{U}})$ is a GT_0-space. Since $(Y, \tau_{\mathcal{U}^*})$ is a reduced extension of $(X, \tau_{\mathcal{U}})$, then by Lemma 4.2 we have $(Y, \tau_{\mathcal{U}^*})$ is a GT_0-space. Again by Proposition 4.2, we get that $(Y, \mathcal{U}^*)$ is separated. □

Now, we introduce the notion of the complete L-uniform space and its completion which is defined similarly as in the classical case by using the Cauchy and convergent filters on uniform spaces but in the setting of fuzzy spaces.

Definition 4.5. An L-uniform space $(X, \mathcal{U})$ is called *complete* if every $\mathcal{U}$-Cauchy filter $\mathcal{M}$ on X is convergent.

Definition 4.6. An L-uniform space $(Y, \mathcal{U}^*)$ is called a *completion* of an L-uniform space $(X, \mathcal{U})$ if it is a reduced extension of $(X, \mathcal{U})$ and $\mathcal{U}^*$ is complete.

Lemma 4.6. *The completion of a separated L-uniform space is separated as well.*

Proof. The result follows from Lemma 4.5. □

5. Completion of L-topological groups

In this section, we introduce main notion of this paper, that completion of L-topological groups.

5.1. L-topological groups

Let G be a multiplicative group. We denote, as usual, the identity element of G by e and the inverse of an element a of G by a^{-1}.

Definition 5.1 (*Ahsanullah [1] and Bayoumi [3]*). Let G be a group and τ an L-topology on G. Then (G, τ) will be called an *L-topological group* if the mappings

$$\pi : (G \times G, \tau \times \tau) \to (G, \tau) \quad \text{defined by } \pi(a, b) = ab \quad \text{for all } a, b \in G$$

and

$$i : (G, \tau) \to (G, \tau) \quad \text{defined by } i(a) = a^{-1} \quad \text{for all } a \in G$$

are L-continuous. π and i are the binary operation and the unary operation of the inverse on G, respectively.

For all $\lambda \in L^G$, denote by λ^i the L-set $\lambda \circ i$ in G, that is, $\lambda^i(x) = \lambda(x^{-1})$ for all $x \in G$. We also denote $\mathcal{F}_L \pi(\mathcal{L} \times \mathcal{M})$ by $\mathcal{L}\mathcal{M}$ and $\mathcal{F}_L i(\mathcal{M})$ by $\mathcal{M}^i$, which means that $\mathcal{L}\mathcal{M}(\lambda) = \mathcal{L} \times \mathcal{M}(\lambda \circ \pi)$ and $\mathcal{M}^i(\lambda) = \mathcal{M}(\lambda^i)$ for all L-filters $\mathcal{L}, \mathcal{M}$ on G and all L-sets $\lambda \in L^G$.

A surrounding $u \in L^{X \times X}$ is called *left* (*right*) *invariant* provided

$$u(ax, ay) = u(x, y), \quad (u(xa, ya) = u(x, y)) \quad \text{for all } a, x, y \in X.$$

$\mathcal{U}$ is called a *left* (*right*) *invariant* L-uniform structure if $\mathcal{U}$ has a valued L-filter base consists of left (right) invariant surroundings [8].

The following proposition introduces the left (right) invariant L-uniform spaces compatible with an L-topological group.

Proposition 5.1 (*Bayoumi and Ibedou [8]*). *Let (G, τ) be an L-topological group. Then there exist on G a unique left invariant L-uniform structure $\mathcal{U}^{\mathrm{l}}$ and a unique right invariant L-uniform structure $\mathcal{U}^{\mathrm{r}}$ compatible with τ, constructed using the family $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$ of all filters α-pr$\,\mathcal{N}(e)$, where $\mathcal{N}(e)$ is the L-neighborhood filter at the identity element e of (G, τ), as follows*:

$$\mathcal{U}^{\mathrm{l}}(u) = \bigvee_{v \in \mathcal{U}^{\mathrm{l}}_\alpha,\, v \leqslant u} \alpha \quad \textit{and} \quad \mathcal{U}^{\mathrm{r}}(u) = \bigvee_{v \in \mathcal{U}^{\mathrm{r}}_\alpha,\, v \leqslant u} \alpha, \tag{5.1}$$

where

$$\mathcal{U}^{\mathrm{l}}_\alpha = \alpha\text{-pr}\,\mathcal{U}^{\mathrm{l}} \quad \textit{and} \quad \mathcal{U}^{\mathrm{r}}_\alpha = \alpha\text{-pr}\,\mathcal{U}^{\mathrm{r}} \tag{5.2}$$

are defined by

$$\mathcal{U}^{\mathrm{l}}_\alpha = \{u \in L^{G \times G} | u(x, y) = (\lambda \wedge \lambda^i)(x^{-1}y) \textit{ for some } \lambda \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{5.3}$$

and

$$\mathcal{U}^{\mathrm{r}}_\alpha = \{u \in L^{G \times G} | u(x, y) = (\lambda \wedge \lambda^i)(xy^{-1}) \textit{ for some } \lambda \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{5.4}$$

We should notice that we shall fix the notations $\mathcal{U}^{\mathrm{l}}, \mathcal{U}^{\mathrm{r}}, \mathcal{U}^{\mathrm{l}}_\alpha$ and $\mathcal{U}^{\mathrm{r}}_\alpha$ along the paper to be these defined above.

Definition 5.2. $\mathcal{U}^{\mathrm{b}} = \mathcal{U}^{\mathrm{l}} \vee \mathcal{U}^{\mathrm{r}}$ is called the *bilateral* L-uniform structure of the L-topological group (G, τ).

Remark 5.1. $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{b}}$-Cauchy filter if it is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter and a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter simultaneously.

Remark 5.2 (*cf. Bayoumi and Ibedou [8]*). For the L-topological group (G, τ), the elements of $\mathcal{U}^{\mathrm{l}}_{\alpha}$ $(\mathcal{U}^{\mathrm{r}}_{\alpha})$ are left (right) invariant surroundings. Moreover, $(\mathcal{U}^{\mathrm{l}}_{\alpha})_{\alpha \in L_0}$ $((\mathcal{U}^{\mathrm{r}}_{\alpha})_{\alpha \in L_0})$ is a valued L-filter base for the left (right) invariant L-uniform structure $\mathcal{U}^{\mathrm{l}}$ $(\mathcal{U}^{\mathrm{r}})$ defined by (5.1)–(5.4), respectively.

An L-filter $\mathcal{M}$ on a set X is called *countable* if the sets α-pr$\mathcal{M}$ are countable for all $\alpha \in L_0$ [9].

Now, suppose that (G, τ) has a countable L-neighborhood filter $\mathcal{N}(e)$ at the identity e. By Proposition 5.1, every L-topological groups is uniformizable, so that the left and the right invariant L-uniform structures $\mathcal{U}^{\mathrm{l}}$ and $\mathcal{U}^{\mathrm{r}}$ has, from Remark 5.2, a countable valued L-filter base $(\mathcal{U}^{\mathrm{l}}_{1/n})_{n \in \mathbf{N}}$ and $(\mathcal{U}^{\mathrm{r}}_{1/n})_{n \in \mathbf{N}}$, respectively.

We may recall that if (G, τ) is an L-topological group and A is a subgroup of G, then the L-topological subspace (A, τ_A) is called an *L-topological subgroup* [3].

Proposition 5.2. *Let (A, τ_A) be an L-topological subgroup of an L-topological group (G, τ), $\mathcal{U}$ a complete L-uniform structure on G compatible with τ, and let $\mathcal{U}_A$ be the L-uniform structure on A compatible with τ_A. Then, we get the following results.*

(d1) *If $\mathcal{L}$ and $\mathcal{M}$ are $\mathcal{U}_A$-Cauchy filters, so is $\mathcal{L}\mathcal{M}$.*
(d2) *If $\mathcal{M}$ is a $\mathcal{U}_A$-Cauchy filter, so is $\mathcal{M}^i$.*

Proof. By Lemma 3.1, $\mathcal{L}$ and $\mathcal{M}$ also are $\mathcal{U}$-Cauchy filters. $\mathcal{U}$ complete implies that $\mathcal{L} \underset{\tau}{\to} x$ and $\mathcal{M} \underset{\tau}{\to} y$ hold for some $x, y \in G$, that is, $\mathcal{L} \leqslant \mathcal{N}(x)$ and $\mathcal{M} \leqslant \mathcal{N}(y)$. Now, for each $\xi \in L^G$ we have

$$\begin{aligned}
\mathcal{L}\mathcal{M}(\xi) &= \mathcal{F}_L \pi(\mathcal{L} \times \mathcal{M})(\xi) \\
&= \mathcal{L} \times \mathcal{M}(\xi \circ \pi) \\
&= \bigvee_{\lambda \times \mu \leqslant \xi \circ \pi} \mathcal{L}(\lambda) \wedge \mathcal{M}(\mu) \\
&\geqslant \bigvee_{\lambda \times \mu \leqslant \xi \circ \pi} \mathcal{N}(x)(\lambda) \wedge \mathcal{N}(y)(\mu) \\
&= \bigvee_{\lambda \times \mu \leqslant \xi \circ \pi} \mathrm{int}_\tau\, \lambda(x) \wedge \mathrm{int}_\tau\, \mu(y) \\
&\geqslant \mathrm{int}_\tau\, \xi(xy) \\
&= \mathcal{N}(xy)(\xi).
\end{aligned}$$

That is, $\mathcal{L}\mathcal{M} \underset{\tau}{\to} xy$ and hence, $\mathcal{L}\mathcal{M}$ is a $\mathcal{U}$-Cauchy filter and hence, from Proposition 3.1 and Lemma 3.1, a $\mathcal{U}_A$-Cauchy filter.

Similarly, each $\mathcal{U}_A$-Cauchy filter $\mathcal{M}$ is a $\mathcal{U}$-Cauchy filter and then $\mathcal{M} \underset{\tau}{\to} x$. By Lemma 2.1, $\mathcal{M}^i(\lambda) = \mathcal{F}_L i(\mathcal{M}) \underset{\tau}{\to} i(x) = x^{-1}$. This means that $\mathcal{M}^i$ is a $\mathcal{U}$-Cauchy filter and also a $\mathcal{U}_A$-Cauchy filter. □

Definition 5.3. An L-uniform structure $\mathcal{U}$ of an L-topological group (G, τ) is said to be *admissible* if $\tau_{\mathcal{U}} = \tau$ and the conditions (d1) and (d2) in Proposition 5.2 are fulfilled.

Definition 5.4. An L-topological group (G, τ) is called *complete* if its bilateral L-uniform structure $\mathcal{U}^{\mathrm{b}}$ is complete. (G, τ) is called *left complete* (*right complete*) if it is complete and its left (right) L-uniform structure $\mathcal{U}^{\mathrm{l}}$ $(\mathcal{U}^{\mathrm{r}})$ is admissible.

Lemma 5.1. *The inverse mapping $i : (G, \tau) \to (G, \tau)$, $i(x) = x^{-1}$, on any L-topological group (G, τ) is $(\mathcal{U}^{\mathrm{l}}, \mathcal{U}^{\mathrm{r}})$-continuous and $(\mathcal{U}^{\mathrm{r}}, \mathcal{U}^{\mathrm{l}})$-continuous. Moreover, $\mathcal{U}^{\mathrm{r}} = \mathcal{F}_L(i \times i)(\mathcal{U}^{\mathrm{l}})$, $\mathcal{U}^{\mathrm{l}} = \mathcal{F}_L(i \times i)(\mathcal{U}^{\mathrm{r}})$.*

Proof. For $u \in \mathcal{U}^{\mathrm{l}}_{\alpha}$ and for some $\lambda \in \alpha$ - pr$\mathcal{N}(e)$, we have

$$(u \circ (i \times i))(x, y) = u(x^{-1}, y^{-1}) = (\lambda \wedge \lambda^{i})(xy^{-1}) = w(x, y)$$

for some $w \in \mathcal{U}^{\mathrm{r}}_{\alpha}$. Since $\mathcal{F}_L(i \times i)(\mathcal{U}^{\mathrm{l}})(u) = \mathcal{U}^{\mathrm{l}}(u \circ (i \times i))$ for all $u \in L^{X\times X}$, then $\mathcal{F}_L(i \times i)(\mathcal{U}^{\mathrm{l}})(u) = \mathcal{U}^{\mathrm{r}}(u)$ for all $u \in L^{X\times X}$, and hence i is a $(\mathcal{U}^{\mathrm{l}}, \mathcal{U}^{\mathrm{r}})$-continuous. Similarly, we get that $\mathcal{F}_L(i \times i)(\mathcal{U}^{\mathrm{r}}) = \mathcal{U}^{\mathrm{l}}$ and i is a $(\mathcal{U}^{\mathrm{r}}, \mathcal{U}^{\mathrm{l}})$-continuous. □

Proposition 5.3. *$\mathcal{M}$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter in an L-topological group (G, τ) if, and only if, $\mathcal{M}^{i}$ is a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter.*

Proof. Since, from Lemma 5.1, the mapping $i : (G, \mathcal{U}^{\mathrm{l}}) \to (G, \mathcal{U}^{\mathrm{r}})$ is $(\mathcal{U}^{\mathrm{l}}, \mathcal{U}^{\mathrm{r}})$-continuous, then $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter which implies, from Lemma 3.2, that $\mathcal{F}_L(i)(\mathcal{M}) = \mathcal{M}^{i}$ is a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter. Similarly, the converse follows. □

Proposition 5.4 (*Gähler et al. [15]*). *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two L-uniform spaces. A mapping $f : (X, \tau_{\mathcal{U}}) \to (Y, \tau_{\mathcal{V}})$ is L-continuous if, and only if, f is $(\mathcal{U}, \mathcal{V})$-continuous.*

We also have this result.

Lemma 5.2. *If $\mathcal{U}$ and $\mathcal{V}$ are two L-uniform structures on an L-topological group (G, τ) and both $\mathcal{L}$ and $\mathcal{M}$ are $\mathcal{U}$-($\mathcal{V}$-) Cauchy filters on G, then $\mathcal{L} \times \mathcal{M}$ is a $\mathcal{U} \times \mathcal{U}$-($\mathcal{V} \times \mathcal{V}$-) Cauchy filter on $G \times G$.*

Proof. From Proposition 2.2, $\mathcal{L} \times \mathcal{M}$ is an L-filter on $G \times G$. Let $\mathcal{L}$ and $\mathcal{M}$ be $\mathcal{U}$-Cauchy filters on G. Then there exist $A, B \subseteq G$ such that $\mathcal{L} \leqslant \dot{A}$ and $\mathcal{M} \leqslant \dot{B}$ and A, B are small of order every surrounding u in $(G, \mathcal{U})$. Now,

$$\begin{aligned}
(\mathcal{L} \times \mathcal{M})(u) &= \bigvee_{\lambda\times\mu\leqslant u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \\
&\geqslant \bigvee_{\lambda\times\mu\leqslant u} (\dot{A}(\lambda) \wedge \dot{B}(\mu)) \\
&= \bigvee_{\lambda\times\mu\leqslant u} \bigwedge_{x\in A,\, y\in B} \lambda(x) \wedge \mu(y) \\
&= \bigvee_{\lambda\times\mu\leqslant u} \bigwedge_{x\in A,\, y\in B} \lambda \times \mu(x, y) \\
&= u(A, B) \\
&= (A \dot{\times} B)(u)
\end{aligned}$$

for all $u \in L^{G\times G}$. That is, there exists $A \times B \subseteq G \times G$ such that $\mathcal{L} \times \mathcal{M} \leqslant (A \dot{\times} B)$.

Let $\psi : (G \times G) \times (G \times G) \to L$ be a mapping and u a surrounding in $(G, \mathcal{U})$. Then from Proposition 5.4, π is $(\mathcal{U} \times \mathcal{U}, \mathcal{U})$-continuous, and thus

$$\alpha \leqslant \mathcal{U}(u) \leqslant \mathcal{F}_L(\pi \times \pi)(\mathcal{U} \times \mathcal{U})(u) = \mathcal{U} \times \mathcal{U}(u \circ (\pi \times \pi)) = \mathcal{U} \times \mathcal{U}(\psi).$$

Also, $u = u^{-1}$ implies that

$$\psi^{-1} = (u \circ (\pi \times \pi))^{-1} = u^{-1} \circ (\pi \times \pi) = u \circ (\pi \times \pi) = \psi,$$

that is, ψ is a surrounding in $(G \times G, \mathcal{U} \times \mathcal{U})$, and for any surrounding ψ in $(G \times G, \mathcal{U} \times \mathcal{U})$, there exists a surrounding u in $(G, \mathcal{U})$ such that $\psi = u \circ (\pi \times \pi)$.

Now, $\alpha \leqslant u(x, y)$ for all $x, y \in A$ and $\beta \leqslant u(r, s)$ for all $r, s \in B$ and for some $\alpha, \beta \in L_0$ imply that $\psi((x, r), (y, s)) = (u \circ (\pi \times \pi))((x, r), (y, s)) = u(xr, ys)$. Choosing $(x, y) = (e, e)$ or $(r, s) = (e, e)$, we get that $u(xr, ys) \geqslant \gamma$ for some

$\gamma \in L_0$. That is, for all $(x, r), (y, s) \in A \times B$, we have $\psi((x, r), (y, s)) \geqslant \gamma$ for some $\gamma \in L_0$. This means that $A \times B$ is small of order every surrounding in $(G \times G, \mathcal{U} \times \mathcal{U})$. Therefore, $\mathcal{L} \times \mathcal{M}$ is a $\mathcal{U} \times \mathcal{U}$-Cauchy filter. □

Proposition 5.5. *If $\mathcal{U}^{\mathrm{l}}$ and $\mathcal{U}^{\mathrm{r}}$ are the left and the right L-uniform structures of an L-topological group (G, τ) and both of $\mathcal{L}$ and $\mathcal{M}$ are $\mathcal{U}^{\mathrm{l}}$-($\mathcal{U}^{\mathrm{r}}$-) Cauchy filters, then $\mathcal{L}\mathcal{M}$ has the same property.*

Proof. From Lemmas 5.2 and 3.2, we have $\mathcal{L}\mathcal{M} = \mathcal{F}_L\pi(\mathcal{L} \times \mathcal{M})$ is a $\mathcal{U}^{\mathrm{l}}$-($\mathcal{U}^{\mathrm{r}}$-) Cauchy filter. □

Accordingly, for $\mathcal{U}^{\mathrm{l}}$ and $\mathcal{U}^{\mathrm{r}}$ the property of being admissible depends on the fact whether condition (d2) of Proposition 5.2 is fulfilled.

The left and the right *L*-uniform structures $\mathcal{U}^{\mathrm{l}}$ and $\mathcal{U}^{\mathrm{r}}$ of an *L*-topological group enjoy the following properties.

Proposition 5.6. *The following statements are equivalent in any L-topological group (G, τ).*

(1) *If $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter, so is $\mathcal{M}^i$;*
(2) *If $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter, so is $\mathcal{M}^i$;*
(3) *Every $\mathcal{U}^{\mathrm{l}}$-Cauchy filter is a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter;*
(4) *Every $\mathcal{U}^{\mathrm{r}}$-Cauchy filter is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter;*
(5) *$\mathcal{U}^{\mathrm{l}}$ is admissible;*
(6) *$\mathcal{U}^{\mathrm{r}}$ is admissible.*

Proof. (1) $\Longleftrightarrow$ (5) and (2) $\Longleftrightarrow$ (6) come from Proposition 5.5.

(1) $\Longleftrightarrow$ (2) follows from Proposition 5.3 and that $(\mathcal{M}^i)^i = \mathcal{M}$.

From (1) if $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter, then $\mathcal{M}^i$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter, and thus $\mathcal{M}$, from Proposition 5.3, is a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter. Hence, (1) $\Longrightarrow$ (3); On the other hand, if (3) is fulfilled, then $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter implies that it is a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter and thus $\mathcal{M}^i$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter. That is, (1) $\Longleftrightarrow$ (3).

(2) $\Longleftrightarrow$ (4) is obtained similarly. □

Proposition 5.7. *If the left L-uniform structure $\mathcal{U}^{\mathrm{l}}$ or the right L-uniform structure $\mathcal{U}^{\mathrm{r}}$ of an L-topological group (G, τ) is complete, then the other one is complete as well and both are admissible.*

Proof. If $\mathcal{U}^{\mathrm{l}}$ is complete and $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{r}}$-Cauchy filter, then from Proposition 5.3, $\mathcal{M}^i$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter, thus $\mathcal{M}^i \underset{\tau}{\to} x$ in *G* and then $\mathcal{M} \underset{\tau}{\to} x^{-1}$. Hence, $\mathcal{U}^{\mathrm{r}}$ is complete, and the completeness of $\mathcal{U}^{\mathrm{l}}$ follows by the same way from the completeness of $\mathcal{U}^{\mathrm{r}}$.

At last, $\mathcal{M}$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter implies that $\mathcal{M}$ converges to $x \in G$, that is, $\mathcal{M} \leqslant \mathcal{U}^{\mathrm{l}}[\dot{x}]$, and then $\mathcal{M}^i \leqslant \mathcal{U}^{\mathrm{l}}[\dot{x^{-1}}]$. From Proposition 3.1, $\mathcal{M}^i$ is a $\mathcal{U}^{\mathrm{l}}$-Cauchy filter. Proposition 5.6 implies that both $\mathcal{U}^{\mathrm{l}}$ and $\mathcal{U}^{\mathrm{r}}$ are admissible. □

Lemma 5.3. *If $\mathcal{U}^{\mathrm{b}}$ is the bilateral L-uniform structure of an L-topological group (G, τ), then i is $(\mathcal{U}^{\mathrm{b}}, \mathcal{U}^{\mathrm{b}})$-continuous.*

Proof. From that $\mathcal{U}^{\mathrm{l}} \leqslant \mathcal{U}^{\mathrm{b}}$ and $\mathcal{U}^{\mathrm{r}} \leqslant \mathcal{U}^{\mathrm{b}}$, we get $\mathcal{F}_L(i \times i)\mathcal{U}^{\mathrm{l}} \leqslant \mathcal{U}^{\mathrm{b}}$ and $\mathcal{F}_L(i \times i)\mathcal{U}^{\mathrm{r}} \leqslant \mathcal{U}^{\mathrm{b}}$ hold, and thus

$$\mathcal{F}_L(i \times i)\mathcal{U}^{\mathrm{b}} = \mathcal{F}_L(i \times i)\mathcal{U}^{\mathrm{l}} \vee \mathcal{F}_L(i \times i)\mathcal{U}^{\mathrm{r}} \leqslant \mathcal{U}^{\mathrm{b}}.$$

Hence, *i* is $(\mathcal{U}^{\mathrm{b}}, \mathcal{U}^{\mathrm{b}})$-continuous. □

5.2. L-metric spaces

L-topological groups fulfill good results related to notions of *L*-pseudometric and *L*-metric spaces. We use here the notion of *L*-metric space defined by means of the notion of *L*-real numbers as defined in [12]. By an *L*-real number it is understood [12] a convex, normal, compactly supported and upper semi-continuous *L*-subset of the set of all real numbers $\mathbf{R}$. The set of all *L*-real numbers is denoted by $\mathbf{R}_L$. $\mathbf{R}$ is canonically embedded into $\mathbf{R}_L$, identifying each real number *a* with the crisp *L*-number $a^{\sim}$ defined by $a^{\sim}(\xi) = 1$ if $\xi = a$ and 0 otherwise. The set of all positive *L*-real numbers is defined and denoted by: $\mathbf{R}_L^* = \{x \in \mathbf{R}_L \mid x(0) = 1 \text{ and } 0^{\sim} \leqslant x\}$ [12].

A mapping $\varrho : X \times X \longrightarrow \mathbf{R}_L^*$ is called an *L-metric* [12] on *X* if the following conditions are fulfilled:

(1) $\varrho(x, y) = 0^\sim$ if and only if $x = y$;
(2) $\varrho(x, y) = \varrho(y, x)$;
(3) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$.

If $\varrho : X \times X \longrightarrow \mathbf{R}_L^*$ satisfied the conditions (2) and (3) and the following condition:

(1)′ $\varrho(x, y) = 0^\sim$ if $x = y$

then it is called an *L-pseudometric* on *X*.

A set *X* equipped with an *L*-pseudometric (*L*-metric) ϱ on *X* is called an *L-pseudometric* (*L-metric*) *space*.

To each *L*-pseudometric (*L*-metric) ϱ on a set *X* is generated canonically a stratified *L*-topology τ_ϱ on *X* which has $\{\varepsilon \circ \varrho_x | \varepsilon \in \mathcal{E},\ x \in X\}$ as a base, where $\varrho_x : X \to \mathbf{R}_L^*$ is the mapping defined by $\varrho_x(y) = \varrho(x, y)$ and

$$\mathcal{E} = \{\overline{\alpha} \wedge R^\delta|_{\mathbf{R}_L^*} | \delta > 0,\ \alpha \in L\} \cup \{\overline{\alpha} | \alpha \in L\},$$

where $\overline{\alpha}$ has $\mathbf{R}_L^*$ as domain.

An *L*-topological space (X, τ) is called *pseudometrizable* (*metrizable*) if there is an *L*-pseudometric (*L*-metric) ϱ on *X* inducing τ, that is, $\tau = \tau_\varrho$.

An *L*-pseudometric ϱ is called *left* (*right*) *invariant* if

$$\varrho(x, y) = \varrho(ax, ay), \quad (\varrho(x, y) = \varrho(xa, ya)) \quad \text{for all } a, x, y \in X.$$

An *L*-topological group (G, τ) is called *separated* if for the identity element *e*, we have $\bigwedge_{\lambda \in \alpha\text{-pr}\mathcal{N}(e)} \lambda(e) \geqslant \alpha$, and $\bigwedge_{\lambda \in \alpha\text{-pr}\mathcal{N}(e)} \lambda(x) < \alpha$ for all $x \in G$ with $x \neq e$ and for all $\alpha \in L_0$ [8].

Now, we have the following result.

Proposition 5.8 (*Bayoumi and Ibedou [9]*). *Let* (G, τ) *be a* (separated) *L-topological group. Then the following statements are equivalent.*

(1) τ *is pseudometrizable* (metrizable);
(2) *e has a countable L-neighborhood filter* $\mathcal{N}(e)$;
(3) τ *can be induced by a left invariant L-pseudometric* (*L*-metric);
(4) τ *can be induced by a right invariant L-pseudometric* (*L*-metric).

Definition 5.5. An *L*-uniform structure $\mathcal{U}$ on a set *X* is called *pseudometrizable* (*metrizable*) if there exists a countable *L*-uniform base for $\mathcal{U}$ (and $\mathcal{U}$ is separated).

Proposition 5.9. *For any* (separated) *L-topological group* (G, τ), *the L-uniform structures* $\mathcal{U}^\mathrm{l}, \mathcal{U}^\mathrm{r}$ *and* $\mathcal{U}^\mathrm{b}$ *constructed in* (5.1)–(5.4) *are pseudometrizable* (metrizable).

Proof. From Proposition 5.8, we have $\tau = \tau_{\varrho_1} = \tau_{\varrho_2}$ where ϱ_1 and ϱ_2 are a left invariant *L*-pseudometric (*L*-metric) and a right invariant *L*-pseudometric (*L*-metric) on *G*, respectively. Hence, $\mathcal{U}_{\varrho_1}$ is left invariant and $\mathcal{U}_{\varrho_2}$ is right invariant. From Proposition 5.1, $\mathcal{U}^\mathrm{l}$ and $\mathcal{U}^\mathrm{r}$ are unique, that is, $\mathcal{U}_{\varrho_1} = \mathcal{U}^\mathrm{l}, \mathcal{U}_{\varrho_2} = \mathcal{U}^\mathrm{r}$ and $\mathcal{U}^\mathrm{l}, \mathcal{U}^\mathrm{r}$ are pseudometrizable (metrizable). Moreover, $\tau_{\mathcal{U}^\mathrm{b}} = \tau_{\mathcal{U}^\mathrm{l} \vee \mathcal{U}^\mathrm{r}} = \tau_{\mathcal{U}^\mathrm{l}} \vee \tau_{\mathcal{U}^\mathrm{r}} = \tau$. Hence, $\mathcal{U}^\mathrm{b}$ is pseudometrizable (metrizable) as well. □

Proposition 5.10 (*Bayoumi [2]*). *Let* $(X, \mathcal{U})$ *be an L-uniform space,* $(A, \mathcal{U}_A)$ *an L-uniform subspace of* $(X, \mathcal{U})$ *and* $(\tau_\mathcal{U})_A$ *the L-subtopology of the L-topology* $\tau_\mathcal{U}$ *associated with* $\mathcal{U}$. *Then the L-topology associated to* $\mathcal{U}_A$ *coincides with* $(\tau_\mathcal{U})_A$, *that is,* $\tau_{(\mathcal{U}_A)} = (\tau_\mathcal{U})_A$.

Lemma 5.4. *Let* (A, τ_A) *be an L-topological subgroup of an L-topological subgroup* (G, τ). *If* $\mathcal{U}^\mathrm{l}, \mathcal{U}^\mathrm{r}$ *and* $\mathcal{U}^\mathrm{b}$ *are the left, the right and the bilateral L-uniform structures of* (G, τ), *then the corresponding L-uniform structures of* (A, τ_A) *are* $(\mathcal{U}^\mathrm{l})_A$, $(\mathcal{U}^\mathrm{r})_A$ *and* $(\mathcal{U}^\mathrm{b})_A$, *respectively.*

Proof. From Proposition 5.10, we have $\tau_{(\mathcal{U}^{\mathrm{l}})_A} = (\tau_{\mathcal{U}^{\mathrm{l}}})_A = \tau_A$ and $\mathcal{U}^{\mathrm{l}}$ and then $(\mathcal{U}^{\mathrm{l}})_A$ is left invariant as well. Hence, $(\mathcal{U}^{\mathrm{l}})_A$ is the left invariant L-uniform structure of (A, τ_A). By the same $(\mathcal{U}^{\mathrm{r}})_A$ is the right invariant L-uniform structure of (A, τ_A) as well. Moreover,

$$\tau_{\mathcal{U}^{\mathrm{b}}_A} = \tau_{(\mathcal{U}^{\mathrm{l}}_A \vee \mathcal{U}^{\mathrm{r}}_A)} = \tau_{\mathcal{U}^{\mathrm{l}}_A} \vee \tau_{\mathcal{U}^{\mathrm{r}}_A} = (\tau_{\mathcal{U}^{\mathrm{l}}})_A \vee (\tau_{\mathcal{U}^{\mathrm{r}}})_A = (\tau_{\mathcal{U}^{\mathrm{b}}})_A = \tau_A. \qquad \square$$

Definition 5.6. A complete separated L-topological (H, σ) is said to be a completion of a separated L-topological group (G, τ) if (G, τ) is a σ-dense subgroup of (H, σ).

To give the essential result in this section, that characterization of the completion of an L-topological group, we need the following results.

Proposition 5.11 (*Bayoumi and Ibedou [8]*). *Let (G, τ) be an L-topological group. Then the following statements are equivalent.*

(1) *The L-topology τ is GT_0.*
(2) *The L-topology τ is GT_2.*
(3) *The L-topological group (G, τ) is separated.*

Proposition 5.12. *Let (G, τ) be a separated L-topological group, $\mathcal{U}$ an admissible L-uniform structure on G, and $(H, \mathcal{V})$ the completion of $(G, \mathcal{U})$. Then an operation $\pi' : H \times H \to H$ can be defined on H in a unique way so that H equipped with π' is a group, and $(H, \tau_{\mathcal{V}})$ is an L-topological group of which (G, τ) is a subgroup.*

Proof. Let $\sigma = \tau_{\mathcal{V}}$. If $\pi' : H \times H \to H$ is defined by $\pi'(y, z) = yz$ for all $y, z \in H$, then $\pi'|_{G \times G} = \pi$. Now, let $\mathcal{L}_x$ and $\mathcal{M}_y$ be two trace filters on H at x and y into H, respectively. Since $\mathcal{L}_x \underset{\sigma}{\to} x$ and $\mathcal{M}_y \underset{\sigma}{\to} y$, that is, $\mathcal{L}_x(\lambda) \geqslant \mathrm{int}_\sigma\, \lambda(x)$ and $\mathcal{M}_y(\mu) \geqslant \mathrm{int}_\sigma\, \mu(y)$, then

$$\begin{aligned}
\mathcal{L}_x \mathcal{M}_y(\xi) &= \mathcal{F}_L \pi'(\mathcal{L}_x \times \mathcal{M}_y)(\xi) \\
&= \mathcal{L}_x \times \mathcal{M}_y(\xi \circ \pi') \\
&= \bigvee_{\lambda \times \mu \leqslant \xi \circ \pi'} \mathcal{L}_x(\lambda) \wedge \mathcal{M}_y(\mu) \\
&\geqslant \bigvee_{\lambda \times \mu \leqslant \xi \circ \pi'} \mathrm{int}_\sigma\, \lambda(x) \wedge \mathrm{int}_\sigma\, \mu(y) \\
&\geqslant \mathrm{int}_\sigma\, \xi(xy) \\
&= \mathcal{N}_\sigma(xy)(\xi)
\end{aligned}$$

and then $\mathcal{L}_x \mathcal{M}_y \underset{\sigma}{\to} xy$. From the fact that $\mathcal{U}$ is separated and from Lemma 4.6 and Proposition 5.11, we get (H, σ) is a GT_2-space. Therefore, these properties, using Lemma 4.3 and Remark 4.2, define π' in a unique way as the only continuous extension of π into $H \times H$. Also, if $i' : H \to H$ is defined by $i'(y) = y^{-1}$ for all $y \in H$, then $i'|_G = i$ and $\mathcal{F}_L i'(\mathcal{L}_x) = \mathcal{L}_x^{i'} \underset{\sigma}{\to} x^{-1}$ for any trace filter $\mathcal{L}_x$ on H, and i' is (σ, σ)-continuous, that is, as in before, i' is a continuous extension of i defined in a unique way.

Since π' is $(\sigma \times \sigma, \sigma)$-continuous and i' is (σ, σ)-continuous, we have (H, σ) is an L-topological group in which (G, τ) is an L-topological subgroup. $\square$

Proposition 5.13. *Under the hypothesis of Proposition* 5.12, *if the left, the right or the bilateral L-uniform structure of $(H, \tau_{\mathcal{U}^*})$ is $\mathcal{U}^{*\mathrm{l}}$, $\mathcal{U}^{*\mathrm{r}}$, or $\mathcal{U}^{*\mathrm{b}}$, respectively, then the corresponding L-uniform structures of (G, τ) is $(\mathcal{U}^{*\mathrm{l}})_G$, $(\mathcal{U}^{*\mathrm{r}})_G$, or $(\mathcal{U}^{*\mathrm{b}})_G$.*

Proof. It is a consequence of Lemma 5.4. $\square$

The following proposition summarizes the results of this paper and give the completion of an L-topological group as characterized in Proposition 5.12.

Proposition 5.14. *Let* (G, τ) *be a separated L-topological group*, $\mathcal{U}^{\mathrm{b}}$ *its bilateral L-uniform structure, and* $(H, \sigma = \tau_{\mathcal{V}})$ *the L-topological group constructed in Proposition* 5.12 *with the choice* $\mathcal{V} = \mathcal{V}^{\mathrm{b}}$. *Then* (H, σ) *is a completion of* (G, τ).

Proof. If $\mathcal{U} = \mathcal{U}^{\mathrm{b}}$, then Proposition 5.12 can be applied and $\mathcal{U}^{\mathrm{b}}$ is admissible since both of $\mathcal{U}^{\mathrm{l}}$ and $\mathcal{U}^{\mathrm{r}}$ are admissible. Also, $\mathcal{V}$ is a complete separated L-uniform structure such that $\sigma = \tau_{\mathcal{V}}$, G is σ-dense in H and $(\mathcal{V})_G = \mathcal{U}^{\mathrm{b}}$. On the other hand, by Proposition 5.13, for the bilateral L-uniform structure $\mathcal{V}^{\mathrm{b}}$ of the L-topological group (H, σ) we have $\sigma = \tau_{(\mathcal{V}^{\mathrm{b}})}$ and $(\mathcal{V}^{\mathrm{b}})_G = \mathcal{U}^{\mathrm{b}}$. Therefore, the bilateral L-uniform structure $\mathcal{V}^{\mathrm{b}}$ of (H, σ) is complete and (H, σ) is a completion of (G, τ). □

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